Kalman-Filter

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Dealing with inaccuracy

- Sensor output in dynamic processes often comes with noise
- Relying on the exact values often creates a fairly inaccurate description
- Tools needed to appropriately deal with noise and extract useful data from sensors

What is the Kalman-Filter?

- Tool to control dynamic processes
- Creates estimates of a system's state based on previous estimates and sensor data
- Wide range of applications
  - Tracking objects
  - Navigation
  - Economics
  - Localization (Robotics)

History

- Named after Rudolf Emil Kálmán, co-inventor
- First described in 1958
- Found one of its first applications in the Apollo program
- Still commonly used for all kinds of navigational tasks

Requirements

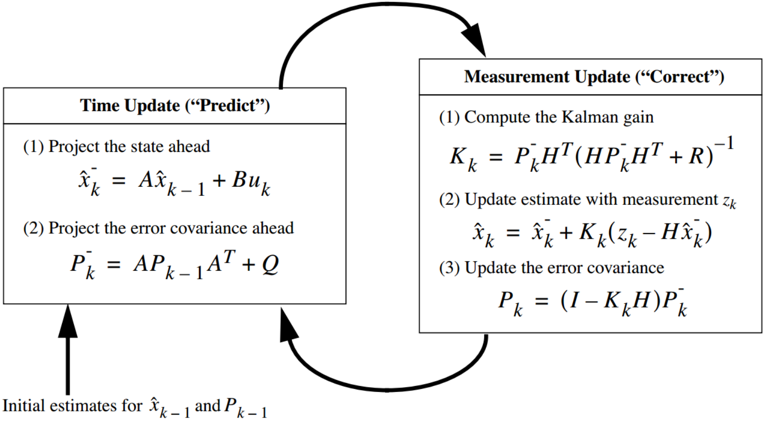
For the most effective usage of the Kalman-Filter the following requirements have to be satisfied:

- Measurements of the system are available at a constant rate
- The error of the measurements follow a gaussian 0-mean distribution
- An accurate model of the process is available

The basic Kalman Filter is limited to linear dependencies between state variables for transitions and measurement.

General principle

- Recursive Algorithm
- Two phases per observation
  - Time Update (Predict)
    - Create a priori estimate of system state based on prior estimation, control input and system dynamics
    - Create a priori estimate of the error covariance matrix
  - Measurement Update (Correct)
    - Compute the Kalman gain, i.e. how strongly the new measurement is factored in for the final estimation
    - Create a posteriori estimate of system state based on a priori estimation, Kalman gain and measurement
    - Update the state error convariance matrix, i.e. the confidence in the new estimation


Time Update ("Predict")
(1) Project the state ahead
$\hat{x}_k^- = A\hat{x}_{k-1} + Bu_k$
(2) Project the error covariance ahead
$P_k^- = AP_{k-1}A^T + Q$
Measurement Update ("Correct")
(1) Compute the Kalman gain
$K_k = P_k^- H^T (HP_k^- H^T + R)^{-1}$
(2) Update estimate with measurement z_k
$\hat{x}_k = \hat{x}_k^- + K_k(z_k - H\hat{x}_k^-)$
(3) Update the error covariance
$P_k = (I - K_k H)P_k^-$
Initial estimates for $\hat{x}_{k-1}$ and P_{k-1}

From [WB95]

Definition

$\hat{x}^-$: A priori estimated state
$\hat{x}$: A posteriori estimated state
A : State transition matrix
B : Control matrix
u : Control input
P^- : State error covariance matrix
Q : Process error covariance matrix
K : Kalman gain
H : Measurement matrix
R : Measurement error covariance matrix
z : Measurement values

Example application

- We observe the firing of a cannonball at a 45° angle
- Four measurement values: Velocities and positions (x & y)
- Measurements are subject to errors (gaussian white noise)
- Goal: Precise estimation of the trajectory of the cannonball

Underlying system dynamics

We obviously know how the laws of physics will affect the cannonball during its flight:

Definition

$$
\begin{aligned}
x_n &= x_{n-1} + V_{x_{n-1}} * \Delta t \\
V_{x_n} &= V_{x_{n-1}} \\
y_n &= y_{n-1} + V_{y_{n-1}} - \frac{1}{2} g \Delta t^2 \\
V_{y_n} &= V_{y_{n-1}} - g \Delta t
\end{aligned}
$$

State transition matrix

Based on these assumptions, we can model the transition matrix A, control Matrix B and control input vector u as follows:

$$A = \begin{pmatrix} 1 & \Delta t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \Delta t \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\Delta t^2 & 0 \\ 0 & 0 & 0 & -\Delta t \end{pmatrix} ; u = \begin{pmatrix} 0 \\ 0 \\ g \\ g \end{pmatrix}$$

Initilization

We initialize the first state estimate with the starting configuration of the system:

$$\hat{x}_0 = \begin{pmatrix} 0 \\ 100cos(\frac{\pi}{4}) \\ 500 \\ 100sin(\frac{\pi}{4}) \end{pmatrix}$$

Note that the initial estimate for y is way off to demonstrate how fast the filter adjusts it.

Initilization

The initial state and process error covariance matrices P and Q are set as:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Initilization

Since our measurement is in actual units, the measurement matrix H is the identity matrix. We assume a certain amount of measurement noise by initializing R as follows:

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; R = \begin{pmatrix} 0.2 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0.2 \end{pmatrix}$$

Prediction

A new state can now be predicted by

$$\hat{x}_n = A\hat{x}_{n-1} + Bu$$

and the estimated covariance matrix follows as

$$P_n^- = AP_{n-1}A^T + Q$$

Correction

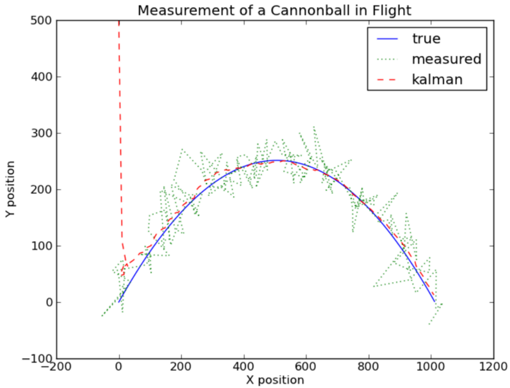
After calculating the optimal Kalman gain, the final estimates for the state and covariance follow in the second phase:

$$K_n = P_n^- H^T (H P_n^- H^T + R)^{-1}$$

$$\hat{x}_n = \hat{x}_n^- + K_n (z_n - H \hat{x}_n^-)$$

$$P_n = (1 - K_n H) P_k^-$$

Result



From [cze]

Extended Kalman-Filter

- Problem with the regular Kalman-Filter: many system or measurement processes are not linear
- The extended Kalman-Filter adresses this problem
- State transition function instead of matrix
- Not an optimal estimator like the linear version, but often reasonable performance
- Standard for navigation systems and GPS

Conclusion

- The Kalman-Filter is a a powerful estimator for dynamic discrete-time systems with process and/or measurement noise
- Provides optimal estimations in the linear case and often good ones in the non-linear case
- Requires a very exact model of the system dynamics to work well

Thank you for your attention!

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